

Mathematics: Solving Equations with a Variety of Operations



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1. Undoing two operations

Think about how to solve this puzzle:

I'm thinking of a number. If I multiply it by 5, and then add 27 to the result, I get 92. What is my number?

To find the number, it is natural to start with the 92 and work backwards, undoing the things I did to it. The last thing I did was to add 27, so before I did that, I must have had 27 less than 92, or $92 - 27 = 65$. That means that my original number times 5 equals 65, so to find that number I can divide 65 by 5 ($= 13$). Click the icon to see how these steps would look using algebra.

There are two operations to undo, so we need two steps. The first step undoes the addition, and replaces it with subtraction; the second undoes the multiplication, and replaces it with division. The order of these steps is important. When the problem was stated in words, there was a clear order of operations: *first* I multiplied by 5, *then* I added 27. Starting from the 92, it makes sense to first undo the step that was done last. The same order is built into the expression $5x + 27$; the rules for order of operations tell us that this means first multiply x by 5, then add 27, so we undo these steps in reverse order. See the Britannica Study Guide *Exponents: An Introduction* for a review of the order of operations.

Think about how you would undo the equation

then click the icon to see the answer.

Notice that even though the 6 appears first in the expression, the order of operations tells us to first divide x by 4, and then add the result to 6. The addition, being the last operation performed, is the first to be undone.

Sometimes it is helpful when determining the order of operations to ask yourself what you would do to evaluate the expression if you knew the value of x . If x was 8, for instance, and you were finding the value of

would you first add 6 to the 8, or would you first divide the 8 by 4?

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2. Undoing many operations

Equations in which the variable only appears once can be solved by undoing, even when they appear to be very complicated. Consider the following equation:

In order to decide what undoing operations need to be done, and in what order to do them, it helps to think of this as if you were stating an "I'm thinking of a number" problem. (See the previous screen "Undoing two operations.") You would start with some number (the x in this case), and then you would tell people what you were doing to that number. You would say the following:

I am thinking of a number. If I ...
 ... multiply it by $\frac{2}{5}$
 subtract 1 from the result
 divide the result by 3
 subtract 6 from the result
 multiply the result by 4
 and add 3 to the result, then I get -17 .

So to find the value of x we need to begin with -17 and undo these operations, going from bottom to top. In other words:

Take -17 , and subtract 3
 divide by 4
 add 6
 multiply by 3
 add 1
 multiply by $\frac{5}{2}$

(Remember that you can divide by $\frac{2}{5}$, or multiply by the reciprocal).

Click this icon to see the resulting sequence of equations.

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3. Preserving equality

Suppose I have a pile of identical blocks of wood, and I challenge you to figure out what each one weighs. I have a balance with two trays, and to give you a clue, I put some of these blocks in each tray, along with blocks that weigh exactly 1 gram each, until the trays balance, as shown by clicking this icon.

As you can see, 5 mystery blocks and 3 one-gram blocks weigh the same as 2 mystery blocks and 7 one-gram blocks. How would you find the weight of each mystery block?

You might see a way to simplify the information given in the balance. Perhaps you see that it would be possible to take 2 mystery blocks from each tray. If you take the same quantity from both trays, the balance should be preserved. The advantage of doing this is that you can now see that 3 mystery blocks plus 3 one-gram blocks weigh the same as 7 one-gram blocks, as shown by clicking the icon.

That tells you that 3 mystery blocks weigh 4 grams. To see this you can take 3 one-gram blocks out of each tray. Now we know that each mystery block weighs $\frac{1}{3}$ of 4 grams, or $\frac{4}{3}$ grams.

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4. Operating on both sides of an equation

If we let x represent the weight of each mystery block in the puzzle on the previous screen "Preserving equality," then we can represent the initial clue with the equation $5x + 3 = 2x + 7$, where each side of the equation represents one of the trays, and the equal sign tells us that they balance. After the first removal of blocks, the equation would be $3x + 3 = 7$. The next removal leaves $3x = 4$, which leads to the final result:

Let's compare each equation to the next one.

Algebraically, we could get from the first equation to the second by taking away $2x$ from each side, and from the second to the third by taking away 3 from each side. Finally, we see that each side of the last equation

is $1/3$ as much as the previous equation $3x = 4$, so we can get there by dividing each side by three. This is a powerful method for solving more complicated equations. An equation is like a balance; if we do the same operation to both sides, the balance is preserved and the resulting expressions are also equal. Since there are an infinite number of possibilities, how do you decide what to do to both sides to make a new equation? The new equation will be useful if it moves us toward having all the x 's (or whatever the variable is) on one side of the equation, and terms consisting only of known numbers, called constant terms, on the other side.

There is more than one sequence of transformations that will lead to the answer. After awhile, you will recognize which transformations are useful, and which make the next steps easier. That is part of the art of solving an equation. If after transforming an equation you find that the result is more complicated or less useful than the previous equation, then go back to that equation and try a different transformation.

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5. Checking answers

One nice thing about solving equations is that you can always tell whether you've made a mistake. To see if your answer really does solve the equation, go back to the original equation (make sure it is the original; the mistake may have been in the very first step!) and replace each variable with the number you think is the solution. Then simplify both sides of the equation; if the resulting statement is true, your solution was correct. If not, you made a mistake somewhere, and you need to go back and check your steps. If you can't find a mistake in your solution, don't forget to check your check! Make sure that you substituted and simplified correctly in your check.

To check that the solution to the equation in the previous screen, $5x + 3 = 2x + 7$, really is $x = 4/3$, we replace each x by $4/3$, inserting parentheses to show the multiplication:

Simplifying gives us

or

Since this is true, the solution was correct.

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6. A helpful equation-solving example

Consider the equation $43 - 5x = 31$. Like most problems in math, there is more than one way to solve it. First, we will transform the equation by doing the same thing to both sides. Our goal is to isolate the term containing x , so we would like to get rid of the 43. Recall that $43 - 5x$ means the same thing as $43 + -5x$, so if we subtract 43 from both sides, we will be left with $-5x = 31 - 43$.

To keep track of what we have done, it's helpful to show the transformation; one way is to write what we are doing to both sides below the equation and then show the result (as shown when you click this icon).

Next we can divide both sides by -5 , or simply replace the multiplication with the equivalent division, to get the solution

$$x = -12 \div -5 = 2.4$$

We could also have solved this by undoing.

So $5x = 12$ and $x = 2.4$. Compare the first and third equations. The first says that taking $5x$ away from 43 leaves 31; the third says that taking 31 away from 43 leaves $5x$. That makes sense. The sum of the parts equals the whole, and the whole minus one part equals the other part. As the second equation reminds us, the whole is 43, and the parts are 31 and $5x$, so when either part is taken away from 43, the other part will be what's left. In general, if $a - b = c$, then $a - c = b$.

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7. Another helpful example

This time we will solve the equation $29 - 8x = 2x + 1 + 4x$.

To begin with, notice that the expression on the right side of the equation is not in its simplest form. You should always check to see if either side can be simplified, before beginning to transform the equation. By combining similar terms, we get $29 - 8x = 6x + 1$. Our goal will be to eliminate the x term from one side, and eliminate the constant term from the other side. To eliminate one of the x terms, we have two choices: we can subtract $6x$ from both sides, or we can add $8x$ to both sides. Both will work, but adding $8x$ will result in the remaining x term having a positive number multiplying the x (this number is called a coefficient), which most people prefer.

Click the icon to see how this looks.

From here it's easy to finish by undoing: $14x = 29 - 1 = 28$, so $x = 2$.

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8. Equations containing parentheses

Sometimes parentheses just change the order in which we operate, and sometimes they require extra steps. We will look at both cases.

Example 1

Think about the equation $26(x + 41) = 52$. Many people automatically apply the distributive property to remove the parentheses, but that just creates a lot of extra work! This equation says "if you add 41 to x , and then multiply the result by 26, you get 52." This is easy to solve by undoing, and this time we undo the multiplication first, since it is the last thing that happens to make the 52. So first we replace the multiplication by an equivalent division statement (or divide both sides by 26, if you prefer to visualize it that way), then we undo the addition. Click the icon to see this.

Example 2

Now consider the equation $7(x - 2) = 3(2x - 6)$. If we approached this example like the previous example, by dividing by 7 or by 3, we would very quickly have some ugly fractions to contend with. We need to consolidate the x terms on one side, but we can't yet. We don't just have x on one side and $2x$ on the other; we have 7 groups containing x and 3 groups containing $2x$.

In general, you don't want to start transforming the equation while it still contains parentheses, except for situations such as Example 1, where x appears only once and you can isolate it by undoing. If you are not sure, the safest thing to do is to get rid of the parentheses by using the distributive law. Here, this is the best strategy. After performing the distributive law to each side, we will have an equation of the kind that we solved in the screen "Another helpful example," and we can use familiar transformations to solve it. Click the icon to see this.

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9. Pitfalls involving parentheses

Equations with many parentheses and terms can be very confusing. At some point, you may have been asked to solve a problem like the following:

In my pocket I have half as many quarters as nickels, and four more nickels than dimes. These coins have a total value of \$2.35, how many of each coin do I have?

If we let x represent the number of nickels, then $x/2$ is the number of quarters and $x - 4$ is the number of dimes. To express the value of the coins we must multiply the number of coins by the value of each, so the value of the quarters, in cents (to avoid decimals), is

the value of the dimes in cents is $10(x - 4)$, and the value of the nickels is $5x$.

Since the total value of the coins is \$2.35, which is 235 cents, we can write that as

Because x appears more than once, we cannot begin by undoing. Sometimes students are tempted to add 4 to each side, but because the -4 is in parentheses which are multiplied by 10, adding 4 to each side will not cancel the subtraction. Instead we must use the distributive law to rewrite the equation without parentheses. This yields:

$$12.5x + 10x - 40 + 5x = 235$$

A common mistake here is to notice the three terms containing x , and to begin trying to consolidate them by subtracting something from both sides. What you must notice, however, is that all the terms containing x are on the same side of the equals sign! That means that the way to consolidate the x terms is to simplify the left side of the equation by combining similar terms. This yields:

$$27.5x - 40 = 235$$

This is easily solved, so the moral of the story is simplify, simplify, simplify. When both sides of an equation are fully simplified, then work on solving the equation with the techniques discussed in this guide. In this case, you can solve by undoing.

$$27.5x = 235 + 40 = 275, \text{ so } x = 275 \div 27.5 = 10.$$

To check we go back to the original problem and substitute our answer of 10 nickels. If there are 10 nickels, worth \$.50, there must be 5 quarters, worth \$1.25, and 6 dimes, worth \$.60, for a total of \$2.35, as desired.

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Glossary

coefficient

the number which multiplies a variable or a product of variables

constant terms

numbers which are added to or subtracted from an expression. A *term* is something added or subtracted, and *constant* means not containing any variables.

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Teacher's Notes

This unit deals with multi-step equation solving. We introduce the familiar technique of "doing the same thing to both sides," visualized as balancing weights which can be manipulated as long as the balance is preserved. At the same time, we continue encouraging students to read the language of algebra by emphasizing the "undoing" approach wherever possible. This is vital, because too many students manipulate symbols based on superficial patterns, without regard to their meaning, and those are the students who cannot generalize and adapt what they've learned to new situations. We want students to be aware that many different sequences of transformations can be used to solve a given equation, so that they can begin to make strategic choices rather than do the first thing that comes to mind. The final screens are intended to help students prevent some common errors.

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Help

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