Mathematics: Inequalities in One and Two Variables

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1. Introduction to inequalities

Imagine you own a store that sells pots and pans. Your markup for pots is $5 and for pans it is $8. The cost of running your store is $100 per week. How many pots and pans do you need to sell to make a profit? A mathematician would write this problem as an inequality:

where *x* is the number of pots sold and *y* the number of pans. There are several ways you can solve inequalities—using guess and check, graphing, or equation solving.

Solutions to inequalities involve defining sets of numbers rather than specific values. This is because there will usually be a whole range of possible solutions to an inequality.

Click the icon to see the solution set to the pots and pans inequality. We will explain how you can do this in the following screens.

The solutions might include all real numbers within that range, or all rational numbers, or perhaps only integers. In the pots and pans example the solution will be positive integers—you can only sell whole pots and pans! If you don't remember how these number sets are defined, click them to view their definitions. Solutions to inequalities can be written symbolically or shown on a graph.

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2. Solving one-variable inequalities

You can solve inequalities using similar techniques to those you use for solving equations, except for one very important exception. When you multiply or divide both sides of an inequality by a negative number, the direction of the inequality sign must be reversed. This must be done to retain the truth-value of the expression. It is worthwhile to check this with numbers. It is true that $3 < 5$. But when you multiply or divide both sides by –1, you get $-3 < -5$, which is not true! However, if you reverse the direction of the inequality sign, then $-3 > -5$ is true. To review solving equations, refer to the Britannica Study Guides on solving equations.

Example 1

$2x - 5 < 7$

Add 5 to both sides:

$2x < 12$

Divide both sides by 2:

$x < 6$

This may also be displayed on a number line, as seen by clicking the icon.

Since it is easy to forget the sign reversal mentioned above, it is critical to check your final solution by testing the original inequality and making certain that numbers in your solution set actually work! In example 1, substituting 6 in the original inequality shows that you found the number that makes both sides equal, since 2(6) – 5 is 7. This shows that you found the correct number to define the "boundary" of your solution set. Testing a number in your solution set that is not on the boundary allows you to determine whether or not you have the sign pointing in the right direction. In the first example, 5 and zero are in the solution set. Either one makes the original inequality true upon substitution. For example, 2(0) – 5 is –5 which is less than 7. Checking this way is a very useful tool for multiple choice tests.

Example 2

Combine like terms:

Multiply both sides by 20/9:

When the inequality sign has an "or equal to" component, the circle on the number line is solid to show that the boundary point of the range of solutions is included in the solution set.

Links:
A review of solving equations.

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3. Solving one-variable inequalities: more examples

Here are some more examples of worked solutions to inequalities in one variable. Remember to check your solution sets by testing the boundary point, and at least one other number.

Example 3
Remove fractions by multiplying the lowest common denominator across the inequality sign. There is no sign reversal because in this case the number is positive.

Multiply both sides by 4:

$3x - 1 > 14 - 2x$

Add $2x$ to both sides:

$5x - 1 > 14$

Add 1 to both sides:

$5x > 15$

Divide both sides by 5:

$x > 3$

Example 4

Multiply both sides by 5:

$2 - 3x < 10x + 15$

Subtract $10x$ from both sides:

$2 - 13x < 15$

Subtract 2 from both sides:

$-13x < 13$

Divide both sides by –13 and change the direction of the sign:

$x > -1$

If you had not reversed the inequality sign, you would have shown that numbers less than –1 should work. A check of some numbers less than –1 (try –2) in the original inequality would have shown that these do not work.

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4. Finite solution sets in inequalities

You may have noticed that all of the solution sets to the questions in the previous activity were infinite sets of numbers with either an upper bound or a lower bound. Sometimes the solution set of an inequality may be restricted to a finite set of values. Let us look at some examples of this type.

Consider the inequality

This requires that we find the solutions to the inequality, which are in the set of integers between 5 and 9 inclusive. Thus, we need to look within a restricted set for any possible solutions.

Note that the

symbol means "is an element of."

The solution we get to the inequality if we ignore the restricted solution set is $x < 7$.

However, we are told in the question that the solution can only include integers from 5 to 9. The only numbers that work from the set of possible values that x can take are 5 and 6. Therefore, our solution to the inequality is $x = 5$ or 6. Click the icon to see this solution set on a number line.

Notice that guess and check is an efficient method for this problem, since there are so few numbers to check!

In summary, if the solution set is to be a restricted set, you can work out the solution by ignoring the restriction, and then determine the intersection of this solution with the restricted set. If the restricted set of possible values is very small, it may be more efficient to simply test the values directly in the inequality.

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5. Solution sets in inequalities with upper and lower bounds

Let us examine an inequality such as

There are two inequality signs here. This usually means you have a lower and an upper bound for your solution set. The possible values for *x* have not been restricted as they were in the previous example, so we are working within the entire set of real numbers.

An efficient way to solve an inequality of this type is to work on all three "parts" at the same time. By each part, we mean each expression that is separated by an inequality sign. Focus on your usual techniques for isolating the variable, but do your inverse operations to each of the three parts! This is how you might go about doing it.

Add 1 to all parts

Divide all parts by 2:

This solution actually reads as: "2 is less than or equal to *x*, which is less than 7," or "*x* is greater than or equal to 2, and less than 7."

Again, this can be easily represented on a number line and checked by testing numbers.

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6. Some applications of one-variable inequalities

Sometimes worded statements can be expressed as inequalities and solved to find a solution to a practical problem, as shown in the following example.

Example: If 12 is subtracted from four times a certain integer and the result is divided by three, the answer is less than 4. What are the possible values of the integer?

We begin by making an inequality statement from this information. Let *x* represent the unknown number. Translating the English sentence into a math sentence results in the statement:

Look at the English sentence and see if you think our math sentence correctly represents what is being said. If we then simplify this inequality, the solution can be written as

the set of integers.

This means that in order for the statement to be satisfied, the unknown value must be an integer less than 6. Check for yourself that this is true. Remember to first check 6 and see if both sides are equal. Then check a number that is in your solution set, to make sure you have the sign pointing in the right direction. In this case, 0 is in your solution set, and 0 is often an easy number to use. Substituting 0 results in the statement –4 < 4, which is true.

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7. Solving two-variable inequalities

Solutions to inequalities with two variables are best shown graphically. In many cases, the boundary of the solution set will be a line.

A graph of a linear equation divides a Cartesian plane into three regions: the points on the line, those on one side of the line, and those on the other side of the line, as seen by clicking the icon.

The straight line in the diagram is the graph of the linear equation $y = x + 3$. It shows all of the points that make the equation $y = x + 3$ true.

The region above the line, shaded in red, represents points whose x and y values make the inequality $y > x + 3$ true. The region below the line, shaded in blue, represents points whose x and y values make $y < x + 3$ true.

The solution to a two-variable inequality, then, is a region of points on a plane. As with the one-variable, or one-dimensional, case discussed earlier in this guide, there are three questions to consider in determining the solution. What is the boundary of the region, on what side of the boundary are the solution points, and should the boundary be included in the solution?

- As with the one-variable case, to find the boundary of the set, look for the solutions that make both sides of the inequality equal. In other words, change the inequality sign to an equal sign and graph the resulting linear equation—this will be your boundary line!
- Since the boundary line represents points that make both sides equal, they should be included in the solution set region only if the inequality includes an equality, as in

This is shown graphically by a solid boundary line.

- If the signs used in the inequality are < or >, then an equality is not included and the points on the boundary line should not be included in the solution set. This is shown graphically by making a dashed boundary line instead of a solid line.
- Finally, you need to determine which side of the line should be shaded. Simply pick any point that is not on the line, and see if that point's x and y values make the inequality true. The point (0,0) is usually an easy point to evaluate. If the x and y values make the inequality true, then that point is in the solution set, and so are all of the points on the same side of the boundary line, and you should shade them. If they make the inequality false, then that point is not in the solution set, and the points on the other side of the line should be shaded.

The regions defined above are commonly known as half planes, and are closed if the line is solid, and open when the line is broken.

To sketch the graphical solution of a two-variable inequality you need to be able to:

- sketch the graph of a linear equation;
- identify the true and false regions of the graph.

The following screen gives a worked example of this method.

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8. Applications of two-variable inequalities

Graphing two-variable inequalities is a way of showing the large number of solutions that can satisfy an inequality. In an application, the inequality is sometimes called a constraint. Imagine selling artistic refrigerator magnets at a craft fair. You sell small ones for $2 each and large ones for $5 each. You have to pay the craft fair $20 for the hire of your stall, and you obviously want to do better than just break even for the day. This can be expressed as the inequality

$2S + 5L > 20$, where S = the number of small refrigerator magnets sold and L = the number of large refrigerator magnets sold.

This "constrains" the possible solutions. Some additional constraints that people would assume for this problem are that S,L are greater than or equal to zero and that S and L are in the set of integers (since you generally do not sell partial magnets).

We can show the solution set of $2S + 5L > 20$ on a graph, and then bring in the other restrictions. The process is shown below.

Let $y = S$ and $x = L$. It does not matter which is which.

1. Let $2y + 5x = 20$. (Determine the boundary of the solution set by looking at the related linear equation.)

2. Sketch the graph of this line using a dashed line (if you need to review how to graph a line, see Britannica Study Guides on graphing). The line is dashed because it represents the "strictly greater than" situation to show that we are unwilling to accept simply "breaking even" with our sales for the day. It would be solid if it was acceptable for the money from the sale of refrigerator magnets to be equal to 20.

3. Identify the region that contains the range of possible solutions by testing ordered pairs that are not on the line. The whole region above the line satisfies the inequality. Click the icon to see this region.

4. We shade the region that satisfies the inequality, and hence the region above the line is shaded. This area shows points that represent making a profit. However, we should bring in the other assumed conditions, namely that

and that x, y are in the set of integers. Click the icon to see the intersection of all three of these sets.

Intersection will be explained in more detail in the screen on intersections.

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9. Graphing intersections of solution sets

We can graph two or more inequalities on one set of axes, and by doing so find the intersection of the individual solution sets or regions. An intersection can be described mathematically by the symbol

Consider the following mathematical statement:

Find the region corresponding to

This appears to be quite a complex problem, but on closer examination you will find that it is not as difficult as it may seem. What the problem is actually asking for is the intersection (or the common region) of the two solution sets of the given inequalities. This can be shown graphically.

Since we have been working in the form where *y* is alone on one side of the inequality sign, we will first put both inequalities in this form.

The inequality

can be written as

and

requires no rearranging. Therefore, we have two relatively simple inequalities to solve graphically.

First we replace the inequality signs by equal signs, graph the lines, and sketch the graphs. Remember to use a dotted line for $y > x + 2$ as it is does not include the equality condition.

Next, we need to shade the appropriate regions for each inequality. Remember to shade the region that represents points that make the inequality true. To review your skills, try to do all of this work yourself before clicking the icon to see the solution.

Parts of the graph are shaded red, blue, and purple, and one part is not shaded at all (the white part). It is the region in purple, the area that represents the intersection of the blue and red regions, that is our solution set. Any set of points (x, y) in this region will have x and y values that satisfy both of these inequalities.

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Glossary

Cartesian plane
a plane organized along two perpendicular coordinate axes, typically labeled *x* on the horizontal axis and *y* on the vertical axis. All points on the plane can be located by using an *x*-coordinate and a *y*-coordinate. These points are written (*x*, *y*).

constraint
a condition or restriction that limits the range of possible solutions from a universal (or infinite) set to a smaller defined set

element of a set
an object of a set

half plane
set of points (*x*, *y*) on one side of the line of a linear equation as defined by a two-variable inequality. If the inequality includes the line, it is a closed half plane; if it does not, it is an open half plane.

inequality
a statement in one of the following forms

where *A* and *B* are numbers or algebraic expressions

integers
the set of whole numbers {0, 1, 2, 3,...} or their opposite {...,–3, –2, –1, 0} sometimes this is shown as {..., –2, –1, 0, 1, 2, ...}

intercept
the point where the graph of the equation crosses either the *x*-axis (*x*-intercept) or the *y*-axis (*y*-intercept)

intersection
the intersection of two sets *X* and *Y* is the set consisting of all elements that belong to both set *X* and set *Y*.

rational numbers
numbers that can be expressed as a ratio or fraction such as *n*/*m* where *n* and *m* are integers, with *m* not equal to zero

real numbers
the set of all numbers, rational and irrational, such as
–3, 2/7, , *e*, and the square root of 5.

Real numbers do not include complex or imaginary numbers like the square root of –1.

set
a collection of objects or values. The set can be defined by listing its elements. For example, the set of numbers on a die is the set {1,2,3,4,5,6}.

y = mx + b
the standard form of a linear equation where (x, y) is the general form of any ordered pair satisfying the equation, m is the slope, and b is the y-intercept

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Teacher's Notes

This study guide is written for students who have a good grounding in basic algebraic skills, are able to solve linear equations, and have a basic knowledge of set notation.

Although we give some instruction on these skills within this guide, we emphasize the ways in which these same skills can be used in a slightly modified form to solve inequalities. We begin with a review of single-variable inequalities, and then we introduce two-variable inequalities.

We try to help students move beyond solving and graphing linear equations to appreciating that there are vast sets of solutions associated with inequalities and that we can most easily see these solutions graphically. We introduce the concept of half planes, which define many possible solutions through practical, real-life interpretations.

The connection between mathematical logic and graphs can thus be seen in a new light. Solving an inequality basically asks, "What makes the statement true?" When they see an inequality from this perspective, students are encouraged to test solutions from a large set. Hopefully, they will realize that a graph is a "picture" of the set of points that make the statement true.

Students will be able to understand the concepts through simple examples, such as selling goods to recover a fixed cost. We also show the difference between continuous and integer solutions.

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Help

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Links

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Glossary

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